

## Abductive and Inductive Arguments as Varieties of Generalized Division Problems

*"The mind is forced by the very nature of inference itself to make use of induction and hypothesis [abduction]."*

C.S. Peirce  
*The Logic of Science or Induction and Hypothesis*  
Lowell Lectures of 1866

*"Philosophy ought to imitate the successful sciences in its methods ... and to trust rather to the multitude and variety of its arguments than to the conclusiveness of any one."*

C.S. Peirce  
*Some Consequences of Four Incapacities*

### I. Introduction

Although some philosophers and logicians prior to Charles Sanders Peirce acknowledged that, besides deductions, there are other kinds of reasoning, he is the first to present an organized account of the differences among deductive, inductive, and abductive arguments, that these three major classes of argument are necessary, and how they are interrelated as facets of a systematic whole. In this essay I will first summarize both his central contention that inductions and abductions are permutations on deductions as applicable to Aristotelian syllogisms. Next I will present a new account of the abduction/deduction/induction trichotomy based on the recognition, first due to Augustus de Morgan, that a deductive syllogism involves the binary operation of the relational composition of its two premisses yielding a conclusion. It is a commonplace in universal algebra that every binary operation has two generalized division problems, a left division problem and a right division problem. This account will utilize concepts and tools from abstract algebra and category theory yielding results that match with the permutational account of abductive and inductive syllogisms. Further elucidation and justification will be exemplified by employing Fred Sommers' Term Logic. While the former account is a syntactical account, the latter will also supply the preliminaries of a semantical account.

### II. Peirce's Permutational Account of the Trichotomy of the Major Classes of Syllogism

Peirce, like Aristotle, conceived of arguments as a marriage of form and matter. Further, he believed that form should be the principal criterion of classification of arguments. Moreover, he regarded form as mathematical<sup>i</sup>. Peirce demonstrated that there are three and only three major classes of argument, any other sorts of arguments being compounds of any two or all three of these fundamental forms<sup>ii</sup>.

Standard deductive syllogisms are simple arguments, that is, arguments consisting of three component two-term categorical propositions, two as premisses and the remaining one as conclusion. Peirce, following Immanuel Kant, regarded the major premiss as a Rule; the minor premiss provides a Case subsumed under that Rule; and the subsumption of that Case under that Rule yields a Result as the conclusion<sup>iii</sup>. Thus, every deductive syllogism satisfies the general schema in Figure 1.

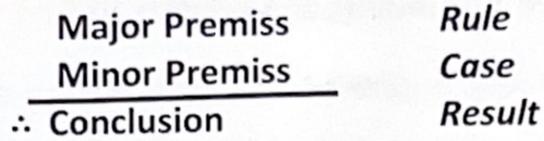


Figure 1: General Form of Deductive Syllogism

Peirce claimed that while a deductive syllogism involves a Rule together with a Case necessarily leading to a Result, an abductive syllogism involves a Rule and a Result yielding a Case and an inductive syllogism involves a Result and a Case yielding a Rule<sup>iv</sup>. He recognized that for each deductive syllogism there is both a counterpart abductive and a counterpart inductive syllogism each of which are produced by a distinct permutation of the constituent propositions of that deductive syllogism. On the one hand, exchanging the Case and the Result of a deductive syllogism produces corresponding the abductive syllogism and on the other hand, exchanging the Rule and the Result of that deductive syllogism produces the corresponding inductive syllogism<sup>v</sup> (Figure 2).

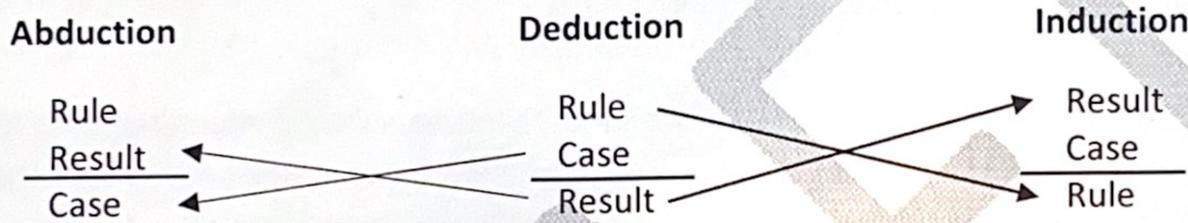


Figure 2: Two Permutations of Simple Deduction

This general procedure can be applied to any deductive syllogism to obtain its abductive and inductive companion syllogisms. If syllogistic is treated as a natural deduction system, only two inference rules would be needed, specifically, the  $AAA_1$  (BARBARA) and the  $AII_1$  (DARII) moods of the syllogisms, provided we permit the use of term complements for the predicates which allow us to express the logical equivalents for *E* and *O* propositions<sup>vi</sup>. Now we can apply the permutational account to  $AAA_1$  and  $AII_1$  syllogisms in order to expand syllogistic to include both abductive and inductive syllogisms (Figure 3).

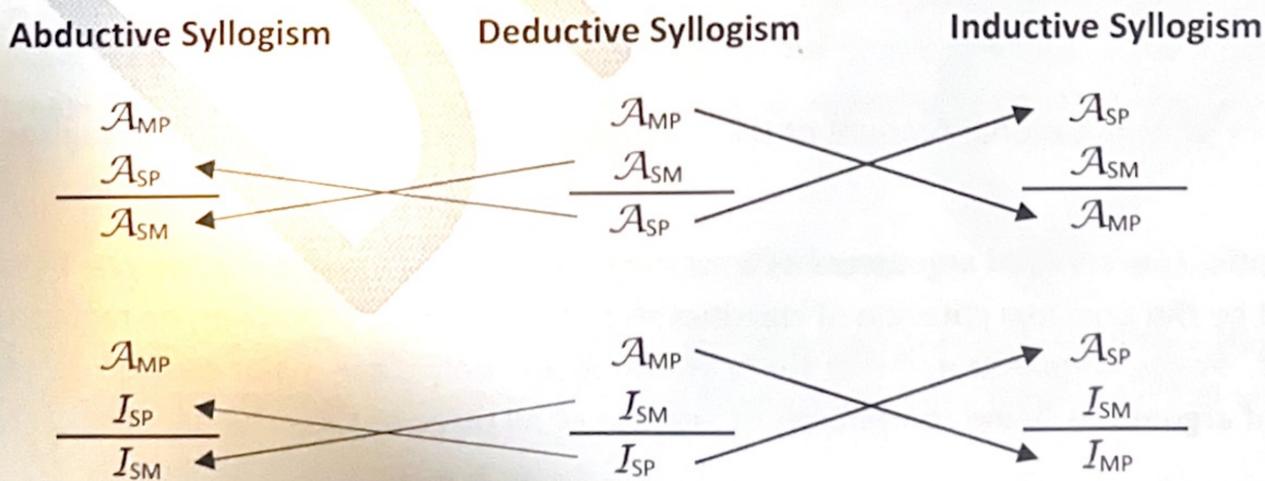


Figure 3: Permutations of  $AAA_1$  and  $AII_1$

If the abductive counterparts to both the deductive  $\mathcal{AAA}_1$  and deductive  $\mathcal{AII}_1$  are assessed deductively, they are instances of the Formal Deductive Fallacy of the Undistributed Middle Term<sup>vii</sup>. This observation has been used to reject abduction as a form of reasoning. These abductive forms, when supplied with appropriate content, would have conclusions containing more information than is contained in their premisses. This is clearly impossible in a deduction since a deduction makes more explicit in its conclusion, what is implicit in its premisses. Deduction's logical task is explication. However, since an abductive syllogism is the means of inferring a hypothesis, this must of necessity be the case, which is to say that abduction's logical task is explanation. Rejecting abduction as a form of reasoning would be like rejecting a pair of pliers as a tool because it cannot be used to drive in nails.

If the inductive counterparts of both the deductive  $\mathcal{AAA}_1$  and deductive  $\mathcal{AII}_1$  are also assessed deductively, the former is an instance of the Formal Deductive Fallacy of the Illicit Minor Term and the latter is an instance of three formal deductive fallacies, those of Two Existential Premisses, the Undistributed Middle Term, and the Illicit Minor Term.

Though less frequently than with abduction, some philosophers have also rejected induction as a form of reasoning, again because a conclusion in such an inference also would contain more information than is contained in its premisses. However, since an inductive syllogism is the means of inferring a generalization, this is also what is needed. Its logical task is extrapolation from cases and so, will of necessity contain more information than what is present in the premisses.

Note the following patterns possessed by abductive, deductive, and inductive syllogisms. Recall that the middle term of a deductive syllogism is the term which occurs twice in the premisses. The middle term of the abductive syllogism is the predicate term of both the major and minor premisses. The middle term of the inductive syllogism is the subject term of both the major and minor premisses. If the propositional forms *All S are P* and *Some S are P* are viewed respectively as a total inclusion relation and a partial inclusion relation with the *S* term as source and the *P* term as target those relations, then the premisses in an abduction share a common predicate and thus, the same target, while the premisses in an induction share a common subject and thus, the same source. In a deduction, however, the middle term is a predicate (target) in the minor premisses and a subject (source) in the major premisses. It is this formal aspect of deduction which is behind Augustus de Morgan's recognition that deductive syllogism can be treated as the relational composition of the major and minor premisses via the middle term<sup>viii</sup>.

### III. Binary Operations and Their Two Division Problems

A binary operation is a mathematical operation composing two operands yielding a single result, that is, where *X*, *Y*, and *Z* are members of some set, ' $\circ$ ' is a binary operation closed over that set, if

$$\forall_x \forall_y \exists_z (X \circ Y = Z)$$

Such quotidian arithmetic operations as multiplication, addition, subtraction, and division are examples of binary operations. Likewise, so are such logical operations as conjunction and alternation.

In both abstract algebra and category theory a binary operation of any sort is usually may referred to as a *multiplication* operation. In other words, *multiplication* is generalized to apply to all binary

mathematical combinations. Every binary operation, that is, every multiplication adduces two canonical\* division problems, a left division problem and a right division problem. Moreover there can only these two division problems. Here this concept of division is as correspondingly general as the abstract algebraic concept of multiplication.

Let's begin to elucidate these concepts of generalized multiplication division by first examining the original multiplication, specifically, arithmetic multiplication over the set of natural numbers  $\mathbb{N} = \{1, 2, \dots, n\}$ . Obviously, it is a binary operation with a multiplier and a multiplicand yielding a single resultant product such that for  $X, Y, Z \in \mathbb{N}$

$$\forall x \forall y \exists z (X \times Y = Z)$$

Two division problems arise naturally for arithmetic multiplication.

*The Left Division Problem for Arithmetic Multiplication:*

For  $\forall y \forall z$ , what are all the answers to  $\lambda$ , if any, such that  $\lambda \times Y = Z$  ?

The solutions, would be obtained by arithmetic division of the form  $\lambda = Z/Y$ . Thus, the kinds of numbers required for the solutions would be positive rational numbers. These solutions lead to an expansion of the concept of number in order for operational closure to hold.

*The Right Division Problem for Arithmetic Multiplication:*

For  $\forall x \forall z$ , what are all the answers to  $\rho$ , if any, such that  $X \times \rho = Z$  ?

The solutions, again obtained by arithmetic division, would be of the form  $\rho = Z/Y$  and would require positive rational numbers in order for operational closure again to hold.

Since arithmetic multiplication is a commutative operation, that is, in an operation such that  $A_x A_y (X \times Y = Y \times X)$ , the left division problem and the right division problem have the same solution sets. However, not every variety of multiplication operation is commutative. In many of those cases the solutions for left division and right division problems could differ. Matrix multiplication is an example of a variety of non-commutative multiplication<sup>x</sup>.

Now, consider the binary operation of arithmetic addition as a variety of generalized multiplication, where  $X, Y, Z \in \mathbb{N}$

$$\forall x \forall y \exists z (X + Y = Z)$$

Again, two division problems arise naturally:

*The Left Division Problem for Arithmetic Addition:*

For  $\forall y \forall z$ , what are all the possible answers to  $\lambda$ , if any, such that  $\lambda + Y = Z$  ?

The solutions, obtained by left arithmetic "division," that is, arithmetic subtraction, would be of the form  $\lambda = -Y + Z$ . Thus, the kinds of numbers required for the solutions would be non-positive whole numbers, that is,  $\{\dots -2, -1, 0\}$ . As with arithmetic multiplication, the solutions lead to an expansion of the concept of number.

*The Right Division Problem for Arithmetic Addition:*

For  $\forall_x \forall_z$ , what are all the possible answers to  $\rho$ , if any, such that  $X + \rho = Z$ ?

The solutions, obtained by right arithmetic "division," that is, again, arithmetic subtraction, would be of the form  $\rho = Z - Y$ . Also, since addition is commutative, the solution sets for both varieties of division would be the same.

Now on to (Generalized) Multiplication by any binary operation. Where  $X, Y, Z$  are members of some set,

$$\forall_x \forall_y \exists_z (X \circ Y = Z)$$

two (generalized) division problems arise naturally<sup>x</sup>.

*The Left Division Problems for Binary Operations:*

For  $\forall_y \forall_z$ , what are all the  $\lambda$ , if any, such that  $\lambda \circ Y = Z$ ?

In category theory this sort of problem is usually called a *determination* problem. In abstract algebra it is called an *extension* problem.

The solutions, if any, would be a variety of left division  $\lambda = Y \setminus Z$ .

*The Right Division Problem for Binary Operations:*

For  $\forall_x \forall_z$ , what are all the  $\rho$  if any, such that  $X \circ \rho = Z$ ?

In category theory this sort of problem may either be called a *choice* problem or, as in abstract algebra, a *lifting* problem.

The solutions, if any, would be a variety of right division  $\rho = Z / X$ .

#### IV. Division Problems for Deductive Syllogisms

Though the first complete formal logic of relations for relations of any adicity was developed by C.S. Peirce, its seeds were in the work of Augustus de Morgan on syllogism.

"In my second and third papers on logic I insisted on the ordinary syllogism being one case, and one case only, of the composition of relations." \* Maddux for citation

As mentioned earlier in this essay, de Morgan correctly saw that categorical propositions can be regarded as dyadic relational propositions whose two relata are the subject and predicate terms related by an inclusion or a partial inclusion relation. A deductive syllogism could then be treated as the composition of its two premisses to yield another two-relata relational proposition. This composition of

relations is itself another binary operation which has come to be called *relative multiplication*. Relative multiplication is not commutative; and so, has two division problems each with a distinct solution.

Since any deductive syllogism involves the binary operation of relative multiplication, it follows that a deductive syllogism also must have two accompanying division problems. These turn out to be inductive and abductive syllogisms.

The Binary Operation of Relational Composition of Categorical Propositions in a Deductive Syllogism

**Rule  $\otimes$  Case = Result** where ' $\otimes$ ' stands for relative multiplication: for all Rules and any Cases falling under those Rules there is some Result.

*The Left Division Problem for Deductive Syllogisms:*

For all Cases and all Results, what are all the  $\lambda$ , if any, such that

$$\lambda \otimes \text{Case} = \text{Result}$$

The solutions, if any, would be a variety of left division  $\lambda = \text{Result} \oslash \text{Case}$

*The Right Division Problem for Deductive Syllogism:*

For all Rules and any Result, what are all the  $\rho$ , if any, such that

$$\text{Rule} \otimes \rho = \text{Case}$$

The solutions, if any, would be a variety of right division  $\rho = \text{Rule} \oslash \text{Case}$

Let's apply this to the two moods of the syllogism used earlier to exemplify Peirce's permutational account of abductive and inductive syllogism, specifically  $AAA_1$  and  $AII_1$ .

$$\begin{array}{l} \mathcal{A}_{MP} \\ \mathcal{A}_{SM} \\ \hline \mathcal{A}_{SP} \end{array} \quad \mathcal{A}_{MP} \otimes \mathcal{A}_{SM} = \mathcal{A}_{SP}$$

$AAA_1$  Deductive Syllogism

Left Division Problem

$$\lambda \otimes \mathcal{A}_{SM} = \mathcal{A}_{SP}$$

$$\begin{array}{r} \mathcal{A}_{SP} \\ \mathcal{A}_{SM} \\ \hline \lambda \end{array}$$

**Solution:**  $\lambda = \mathcal{A}_{MP}$

Inductive Syllogism

Right Division Problem

$$\mathcal{A}_{MP} \otimes \rho = \mathcal{A}_{SP}$$

$$\begin{array}{r} \mathcal{A}_{MP} \\ \mathcal{A}_{SP} \\ \hline \rho \end{array}$$

**Solution:**  $\rho = \mathcal{A}_{SM}$

Abductive Syllogism

$$\frac{\mathcal{A}_{MP}}{I_{SM}} \quad \mathcal{A}_{MP} \otimes I_{SM} = I_{SP}$$

### $\mathcal{A}II_1$ Deductive Syllogism

#### Left Division Problem

$$\lambda \otimes I_{SM} = I_{SP}$$

$$\frac{I_{SP}}{I_{SM}} \\ \lambda$$

**Solution:**  $\lambda = \mathcal{A}_{MP}$

Inductive Syllogism

#### Right Division Problem

$$\mathcal{A}_{MP} \otimes \rho = I_{SP}$$

$$\frac{\mathcal{A}_{MP}}{I_{SP}} \\ \rho$$

**Solution:**  $\rho = I_{SM}$

Abductive Syllogism

The permutational account and the abstract algebraical account of abductive and inductive syllogism agree. The permutational account is a syntactical account since it depends on formal considerations only. The abstract algebraical account involves formal considerations, as it must, also involves a semantic aspect since it tells us what role induction and abduction play in logic; namely, they are two varieties of division problems.

#### V. Deductive, Inductive, and Abductive Syllogisms in Fred Sommers' Term Logic

Syllogistic can be modelled a number of different ways including predicate logic. One of the easiest ways, however, to demonstrate this second analysis of Peirce's trichotomy of major classes of syllogism is to use Fred Sommers' model for Term Logic, since it relies on arithmetic addition and subtraction both in notation and in rules of application. Specifically, his term logic represents the relative multiplication of the premisses in syllogisms by arithmetic addition.

S: Subject term

M: Middle term

P: Predicate term

Unary operations: '-' does double duty as both propositional negation and term complementation

Binary operations: '+' does double duty as both the composition of subject and predicate terms and the composition (relative multiplication) of major and minor premisses.

The four standard forms of two-term categorical propositions are:

$$\mathcal{A}_{SP}: (-S) + (+P) = \overset{\vee}{-S} + P$$

$$\mathcal{E}_{SP}: (-S) + (-P) = \overset{\vee}{-S} - P$$

$$I_{SP}: (+S) + (+P) = \overset{\vee}{+S} + P$$

$$O_{SP}: (+S) + (-P) = \overset{\vee}{+S} - P$$

Hence,  $\mathcal{AAA}_1$  and  $\mathcal{AII}_1$  are straightforwardly represented as summations of two formulas.

$$\begin{array}{r} -M + P \\ -S + M \\ \hline -S + P \end{array}$$

$$\begin{array}{r} -M + P \\ +S + M \\ \hline +S + P \end{array}$$

Drawing a conclusion from the premisses is done by adding the forms for the major and minor premisses together.

This logical addition of two premisses is, of course, a binary operation, and thus, has both a left and a right division problem. The ease with which the solutions can be found is by subtraction from each side of the equation.

*The Left Division Problem for Sommersian  $\mathcal{AAA}_1$*

$$\lambda + (-S + M) = -S + P$$

$$\lambda + (-S + M) - (-S + M) = (-S + P) - (-S + M)$$

$$\lambda = -S + P + S - M$$

$$\lambda = -M + P \text{ as required}$$

*The Right Division Problem for Sommersian  $\mathcal{AAA}_1$*

$$(-M + P) + \rho = -S + P$$

$$-(-M + P) + (-M + P) + \rho = (-M + P) + (-S + P)$$

$$\rho = +M - P - S + P$$

$$\rho = -S + M \text{ as required}$$

The counterpart solutions for left and right division problems  $\mathcal{AII}_1$  are straightforward.

## VI. Conclusion

Recognizing that syllogism involves the binary operation of the relative multiplication of its major and minor premisses not only enriches our understanding of the necessity of inductive and abductive syllogisms as right and left division problems, but also further systematizes these non-monotonic arguments in the context of logic as well as in the wider context of mathematics. This paper is another argument why it would be irrational to reject these forms as varieties of reasoning.

ENDNOTES

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viii Order-theoretic vs. algebraic

ix Matrix multiplication

x William Lawvere

DRAFT